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# Shearlet approximations to the inverse of a family of linear operators

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China**Abstract**

The Radon transform plays an important role in applied mathematics. It is a fundamental problem to reconstruct images from noisy observations of Radon data. Compared with traditional methods, Colonna, Easley and *etc.* apply shearlets to deal with the inverse problem of the Radon transform and receive more effective reconstruction. This paper extends their work to a class of linear operators, which contains Radon, Bessel and Riesz fractional integration transforms as special examples.

**MSC:** 42C15; 42C40**Keywords:** inverse problems; shearlets; approximation; Radon transform; noise

## 1 Introduction and preliminary

The Radon transform is an important tool in medical imaging. Although  $f \in L^1(\mathbb{R}^2)$  can be recovered analytically from the Radon data  $Rf(\theta, t)$ , the solution is unstable and those data are corrupted by some noise in practice [1]. In order to recover the object  $f$  stably and control the amplification of noise in the reconstruction, many methods of regularization were introduced including the Fourier method, singular value decomposition, *etc.* [2]. However, those methods produced a blurred version of the original one.

Curvelets and shearlets were then proposed, which proved to be efficient in dealing with edges [3–7]. In 2002, Candès and Donoho applied curvelets [5] to the inverse problem

$$Y = Rf + \varepsilon W, \quad (1.1)$$

where the recovered function  $f$  is compactly supported and twice continuously differentiable away from a smooth edge;  $W$  denotes a Wiener sheet;  $\varepsilon$  is a noisy level. Because curvelets have complicated structure, Colonna, Easley, *etc.* used shearlets to deal with the problem (1.1) in 2010 and received an effective reconstructive algorithm [8].

Note that the Bessel transform and the Riesz fractional integration transform arise in many scientific areas ranging from physical chemistry to extragalactic astronomy. Then this paper considers a more general problem,

$$Y = Kf + \varepsilon W, \quad (1.2)$$

where  $K$  stands for a linear operator mapping the Hilbert space  $L^2(\mathbb{R}^2)$  to another Hilbert space  $Y$  and satisfies

$$(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha} \hat{f}(\xi) \quad (1.3)$$

with  $b > 0$ ,  $\alpha > 0$  ( $K^*$  is the conjugate operator of  $K$ ). Here and in what follows,  $\hat{f}$  denotes the Fourier transform of  $f$ . The next section shows that Radon, Bessel and Riesz fractional integration transforms satisfy the condition (1.3).

The current paper is organized as follows. Section 2 presents three examples for (1.3) and several lemmas. An approximation result is proved in the last section, which contains Theorem 4.2 of [8] as a special case.

At the end of this section, we introduce some basic knowledge of shearlets, which will be used in our discussions. The Fourier transform of a function  $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$  is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i x \cdot \xi} dx.$$

The classical method extends that definition to  $L^2(\mathbb{R}^2)$  functions.

There exist many different constructions for discrete shearlets. We introduce the construction [8] by taking two functions  $\psi_1, \psi_2$  of one variable such that  $\hat{\psi}_1, \hat{\psi}_2 \in C^\infty(\mathbb{R})$  with their supports  $\text{supp } \hat{\psi}_1 \subset [-\frac{1}{2}, -\frac{1}{16}] \cup [\frac{1}{16}, \frac{1}{2}]$ ,  $\text{supp } \hat{\psi}_2 \subset [-1, 1]$  and

$$\sum_{j \geq 0} |\hat{\psi}_1(2^{-2j}\omega)|^2 = 1 \quad \left(|\omega| \geq \frac{1}{8}\right), \quad \sum_{l=-2^j}^{2^j-1} |\hat{\psi}_2(2^j\omega - l)|^2 = 1 \quad (|\omega| \leq 1).$$

Here,  $C^\infty(\mathbb{R}^n)$  stands for infinitely many times differentiable functions on the Euclidean space  $\mathbb{R}^n$ . Then two shearlet functions  $\psi^{(0)}, \psi^{(1)}$  are defined by

$$\hat{\psi}^{(0)}(\xi) := \hat{\psi}_1(\xi_1) \hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right) \quad \text{and} \quad \hat{\psi}^{(1)}(\xi) := \hat{\psi}_1(\xi_2) \hat{\psi}_2\left(\frac{\xi_1}{\xi_2}\right)$$

respectively.

To introduce discrete shearlets, we need two shear matrices

$$B_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and two dilation matrices

$$A_0 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Define discrete shearlets  $\psi_{j,l,k}^{(d)}(x) := 2^{\frac{3}{2}j} \psi^{(d)}(B_d^l A_d^j x - k)$  for  $j \geq 0$ ,  $-2^j \leq l \leq 2^j - 1$ ,  $k \in \mathbb{Z}^2$  and  $d = 0, 1$ . Then there exists  $\hat{\varphi} \in C_0^\infty(\mathbb{R}^2)$  such that

$$\{\varphi_{j_0,k}(x), \psi_{j,l,k}^{(d)}(x), j \geq j_0 \geq 0, -2^j \leq l \leq 2^j - 1, k \in \mathbb{Z}^2, d = 0, 1\}$$

forms a Parseval frame of  $L^2(\mathbb{R}^2)$ , where  $\varphi_{j_0,k}(x) := 2^{j_0} \varphi(2^{j_0}x - k)$ . More precisely, for  $f \in L^2(\mathbb{R}^2)$ ,

$$f(x) = \sum_{k \in \mathbb{Z}^2} \langle f, \varphi_{j_0,k} \rangle \varphi_{j_0,k}(x) + \sum_{d=0}^1 \sum_{j \geq j_0} \sum_{l=-2^j}^{2^j-1} \sum_{k \in \mathbb{Z}^2} \langle f, \psi_{j,l,k}^{(d)} \rangle \psi_{j,l,k}^{(d)}(x)$$

holds in  $L^2(\mathbb{R}^2)$ . It should be pointed out that  $\psi_{j,l,k}^{(d)}(x)$  are modified for  $l = -2^j$  and  $2^j - 1$ , as seen in [8].

## 2 Examples and lemmas

In this section, we provide three important examples of a linear operator  $K$  satisfying  $(K^* K f)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha} \hat{f}(\xi)$  and present some lemmas which will be used in the next section. To introduce the first one, define a subspace of  $L^2(\mathbb{R}^2)$ ,

$$D(\mathbb{R}^2) := \{f \in L^1(\mathbb{R}^2), f \text{ is bounded}\} \subseteq \left\{f \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} |\xi|^{-1} |\hat{f}(\xi)|^2 d\xi < +\infty\right\}$$

and a Hilbert space

$$L^2([0, \pi) \times \mathbb{R}) := \left\{f(\theta, t), \int_0^\pi \int_{\mathbb{R}} |f(\theta, t)|^2 dt d\theta < +\infty\right\}$$

with the inner product  $\langle f, g \rangle := \int_0^\pi \int_{\mathbb{R}} f(\theta, t) \overline{g(\theta, t)} dt d\theta$ .

**Example 2.1** Let  $L_{\theta,t} := \{(x, y), x \cos \theta + y \sin \theta = t\} \subseteq \mathbb{R}^2$  and  $ds(x, y)$  be the Euclidean measure on the line  $L_{\theta,t}$ . Then the classical Radon transform  $R: D(\mathbb{R}^2) \rightarrow L^2([0, \pi) \times \mathbb{R})$  defined by

$$Rf(\theta, t) = \int_{L_{\theta,t}} f(x, y) ds(x, y)$$

satisfies  $(R^* R f)^\wedge(\xi) = |\xi|^{-1} \hat{f}(\xi)$ .

*Proof* By the definition of  $D(\mathbb{R}^2)$ ,  $\int_{\mathbb{R}^2} |\xi|^{-1} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi < +\infty$  for  $f, g \in D(\mathbb{R}^2)$ . It is easy to see that  $\int_0^{2\pi} d\theta \int_0^{+\infty} \hat{f}(\omega \cos \theta, \omega \sin \theta) \overline{\hat{g}(\omega \cos \theta, \omega \sin \theta)} d\omega = \int_0^\pi d\theta \int_{\mathbb{R}} \hat{f}(\omega \cos \theta, \omega \sin \theta) \times \overline{\hat{g}(\omega \cos \theta, \omega \sin \theta)} d\omega$ . This with the Fourier slice theorem ([1, 9]) and the Plancherel formula leads to

$$\begin{aligned} \int_{\mathbb{R}^2} |\xi|^{-1} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi &= \int_0^{2\pi} d\theta \int_0^{+\infty} \hat{f}(\omega \cos \theta, \omega \sin \theta) \overline{\hat{g}(\omega \cos \theta, \omega \sin \theta)} d\omega \\ &= \int_0^\pi d\theta \int_{\mathbb{R}} (R_\theta f)^\wedge(\omega) \overline{(R_\theta g)^\wedge(\omega)} d\omega \\ &= \int_0^\pi d\theta \int_{\mathbb{R}} Rf(\theta, t) \overline{Rg(\theta, t)} dt = \langle Rf, Rg \rangle, \end{aligned}$$

where  $R_\theta f(t) := Rf(\theta, t)$ . Moreover,  $\langle (R^* R f)^\wedge, \hat{g} \rangle = \langle R^* R f, g \rangle = \langle Rf, Rg \rangle = \langle |\xi|^{-1} \hat{f}(\xi), \hat{g}(\xi) \rangle$  for each  $g \in D(\mathbb{R}^2)$ .

Because  $D(\mathbb{R}^2)$  is dense in  $L^2(\mathbb{R}^2)$ , one receives the desired conclusion  $(R^*Rf)^\wedge(\xi) = |\xi|^{-1}\hat{f}(\xi)$ . Here,  $R^*Rf \in L^2(\mathbb{R}^2)$  for  $f \in D(\mathbb{R}^2)$ . In fact,  $R^*Rf = 4\pi I_1 f$  by [10], where  $I_1$  is the Riesz fractional integration transform defined by

$$I_1(f)(x) := C_\alpha \int_{\mathbb{R}^2} \frac{f(x-y)}{|y|} dy$$

with some normalizing constant  $C_\alpha$  [11]. We rewrite  $I_1(f)(x) = \int_{|y| \leq r} \frac{f(x-y)}{|y|} dy + \int_{|y| > r} \frac{f(x-y)}{|y|} dy =: J_1 + J_2$ . Let  $h(y) = \frac{1}{|y|} 1_{B(0,1)}(y)$ ,  $h_r(y) = \frac{1}{r^2} h(\frac{y}{r})$ , where  $B(0,1)$  stands for the unit ball of  $\mathbb{R}^2$  and  $1_A$  represents an indicator function on the set  $A$ . Then  $J_1 = \int_{|y| \leq r} \frac{f(x-y)}{|y|} dy = r \int_{|y| \leq r} h_r(y) f(x-y) dy \leq r Mf(x)$  by Theorem 9 of reference [12, p.59], where  $Mf$  is the Hardy-Littlewood maximal function of  $f$ .

On the other hand, the Holder inequality implies

$$J_2 \leq \|f\|_{\frac{p}{p-1}} \left( \int_{|y| > r} \frac{1}{|y|^p} dy \right)^{\frac{1}{p}} \leq r^{2-p} \|f\|_{\frac{p}{p-1}}$$

with  $p > 3$ . Take  $r = [Mf(x)]^{-\frac{1}{p-1}}$ , one gets  $I_1(f)(x) \leq [Mf(x)]^{\frac{p-2}{p-1}} (1 + \|f\|_{\frac{p}{p-1}})$  and  $\|I_1(f)\|_2 \leq (1 + \|f\|_{\frac{p}{p-1}}) \|Mf\|_{\frac{2(p-2)}{p-1}}^{\frac{p-2}{p-1}} \lesssim (1 + \|f\|_{\frac{p}{p-1}}) \|f\|_{\frac{2(p-2)}{p-1}}^{\frac{p-2}{p-1}} < +\infty$ , since  $f \in L^{\frac{p}{p-1}}(\mathbb{R}^2) \cap L^{\frac{2(p-2)}{p-1}}(\mathbb{R}^2)$  due to the assumption  $f \in D(\mathbb{R}^2)$  and  $\frac{2(p-2)}{p-1} > 1$ ,  $\frac{p}{p-1} > 1$ .

In order to introduce the next example, we use  $f * g$  to denote the convolution of  $f$  and  $g$ .  $\square$

**Example 2.2** The Bessel operator  $B_\alpha : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$  defined by  $B_\alpha f = b_\alpha * f$  with  $\hat{b}_\alpha(\xi) = (1 + |\xi|^2)^{-\frac{\alpha}{2}}$  and  $\alpha > 0$  satisfies

$$(B_\alpha^* B_\alpha f)^\wedge(\xi) = (1 + |\xi|^2)^{-\alpha} \hat{f}(\xi).$$

*Proof* It is known that  $b_\alpha(x) \in L^1(\mathbb{R}^2)$  for  $\alpha > 0$  [11]. Hence,  $(B_\alpha f)^\wedge(\xi) = \hat{b}_\alpha(\xi) \hat{f}(\xi) = (1 + |\xi|^2)^{-\frac{\alpha}{2}} \hat{f}(\xi)$ . For  $f, g \in L^2(\mathbb{R}^2)$ ,  $\langle (B_\alpha^* B_\alpha f)^\wedge, \hat{g} \rangle = \langle B_\alpha^* B_\alpha f, g \rangle = \langle B_\alpha f, B_\alpha g \rangle = \langle (B_\alpha f)^\wedge, (B_\alpha g)^\wedge \rangle = \langle (1 + |\xi|^2)^{-\alpha} \hat{f}(\xi), \hat{g}(\xi) \rangle$ . Thus,

$$(B_\alpha^* B_\alpha f)^\wedge(\xi) = (1 + |\xi|^2)^{-\alpha} \hat{f}(\xi). \quad \square$$

To introduce the Riesz fractional integration transform, we define

$$D = \{f \in L^2(\mathbb{R}^2), f \text{ has compact support}\} \subseteq L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2).$$

Then  $D \subseteq L^s(\mathbb{R}^2)$  ( $1 \leq s \leq 2$ ). For  $f \in D$  and  $0 < \alpha < 1$ , the Riesz fractional integration transform is defined by

$$I_\alpha(f)(x) := C_\alpha \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{2-\alpha}} dy \in L^2(\mathbb{R}^2), \quad (2.1)$$

where  $C_\alpha$  is the normalizing constant [11]. In order to show  $(I_\alpha^* I_\alpha f)^\wedge(\xi) = |\xi|^{-2\alpha} \hat{f}(\xi)$  for  $f \in D$  and  $0 < \alpha < 1/2$ , we need a lemma ([11], Lemma 2.15).

**Lemma 2.1** Let  $S(\mathbb{R}^2)$  be the Schwartz space and  $\Psi = \{\psi \in S(\mathbb{R}^2), \frac{\partial^\beta}{\partial x^\beta} \psi(0) = 0, \beta \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$  with  $\mathbb{Z}^+$  being the non-negative integer set. Define  $\Phi := \{\varphi = \hat{\psi}, \psi \in \Psi\}$ . Then with  $\alpha > 0$ ,

$$(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha} \hat{f}(\xi)$$

holds for each  $f \in \Phi$ .

**Example 2.3** The transform  $I_\alpha$  defined by (2.1) satisfies  $(I_\alpha^* I_\alpha f)^\wedge(\xi) = |\xi|^{-2\alpha} \hat{f}(\xi)$  for  $f \in D$  and  $0 < \alpha < \frac{1}{2}$ .

*Proof* As proved in Examples 2.1, 2.2, it is sufficient to show that for  $f \in D$ ,

$$(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha} \hat{f}(\xi). \quad (2.2)$$

One proves (2.2) firstly for  $f \in C_0^\infty(\mathbb{R}^2)$ . Take  $\mu(r) \in C^\infty([0, \infty))$  with  $0 \leq \mu(r) \leq 1$  and

$$\mu(r) = \begin{cases} 1, & r \geq 2; \\ 0, & 0 \leq r \leq 1. \end{cases}$$

Define  $\psi_N(\xi) := \mu(N|\xi|)\hat{f}(\xi)$ . Then  $\psi_N(\xi) \in \Psi$  and  $f_N(x) := \check{\psi}_N(x) = \hat{\psi}_N(-x) \in \Phi$ . By Lemma 2.1,

$$(I_\alpha f_N)^\wedge(\xi) = |\xi|^{-\alpha} \hat{f}_N(\xi). \quad (2.3)$$

Let  $k(x)$  be the inverse Fourier transform of the function  $1 - \mu(|x|)$  and  $k_N(x) := \frac{1}{N^2} k(\frac{x}{N})$ . Then  $\int_{\mathbb{R}^2} k(x) dx = 1$  and  $f_N(x) = f(x) - k_N * f(x)$ . Moreover, the classical approximation theorem [11] tells

$$\lim_{N \rightarrow \infty} \|f_N - f\|_p = 0$$

for  $p > 1$ . On the other hand,  $\|I_\alpha f_N - I_\alpha f\|_2 = \|I_\alpha(f_N - f)\|_2 \leq C \|f_N - f\|_{\frac{2}{1+\alpha}}$  due to Theorem 16 [12, p.69]. Hence,  $\lim_{N \rightarrow \infty} \|(I_\alpha f_N)^\wedge(\xi) - (I_\alpha f)^\wedge(\xi)\|_2 = \lim_{N \rightarrow \infty} \|I_\alpha f_N - I_\alpha f\|_2 = 0$ . That is,

$$\lim_{N \rightarrow \infty} (I_\alpha f_N)^\wedge(\xi) = (I_\alpha f)^\wedge(\xi) \quad (2.4)$$

in  $L^2(\mathbb{R}^2)$  sense. Note that  $\| |\xi|^{-\alpha} \hat{f}_N(\xi) - |\xi|^{-\alpha} \hat{f}(\xi) \|_2^2 = \int_{\mathbb{R}^2} |\xi|^{-2\alpha} |\hat{f}(\xi)|^2 [1 - \mu(N|\xi|)]^2 d\xi$ ;  $|\xi|^{-2\alpha} |\hat{f}(\xi)|^2 \in L^1(\mathbb{R}^2)$  with  $0 < \alpha < \frac{1}{2}$  and  $\lim_{N \rightarrow \infty} [1 - \mu(N|\xi|)] = 0$ . Then

$$\lim_{N \rightarrow \infty} \| |\xi|^{-\alpha} \hat{f}_N(\xi) - |\xi|^{-\alpha} \hat{f}(\xi) \|_2 = 0 \quad (2.5)$$

thanks to the Lebesgue dominated convergence theorem, which means  $\lim_{N \rightarrow \infty} |\xi|^{-\alpha} \times \hat{f}_N(\xi) = |\xi|^{-\alpha} \hat{f}(\xi)$  in  $L^2(\mathbb{R}^2)$  sense. This with (2.3), (2.4) shows (2.2) for  $f \in C_0^\infty(\mathbb{R}^2)$ .

In order to show (2.2) for  $f \in D$ , one can find  $g \in C_0^\infty(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} g(x) dx = 1$  and  $\lim_{N \rightarrow \infty} \|f * g_N - f\|_p = 0$  ( $p \geq 1$ ) by Theorem 4.2.1 in [13], where  $g_N(\cdot) = N^2 g(N\cdot)$ . Since

$f * g_N \in C_0^\infty(\mathbb{R}^2)$ , the above proved fact says

$$(I_\alpha(f * g_N))^\wedge(\xi) = |\xi|^{-\alpha}(f * g_N)^\wedge(\xi). \quad (2.6)$$

The same arguments as (2.4) and (2.5) show that  $\lim_{N \rightarrow \infty} (I_\alpha(f * g_N))^\wedge(\xi) = (I_\alpha f)^\wedge(\xi)$  and  $\lim_{N \rightarrow \infty} |\xi|^{-\alpha}(f * g_N)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$ . Hence,

$$(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi).$$

This completes the proof of (2.2) for  $f \in D$ .  $\square$

Next, we prove a lemma which will be used in the next section. For convenience, here and in what follows, we define  $\mathcal{M} = N \cup M$  with  $N = \mathbb{Z}^2$ ,  $M := \{\mu = (j, l, k, d) : j \geq j_0, -2^j \leq l \leq 2^j - 1, k \in \mathbb{Z}^2, d = 0, 1\}$ . Then the shearlet system (introduced in Section 1) can be represented as  $\{s_\mu : \mu \in \mathcal{M}\}$ , where  $s_\mu = \psi_\mu = \psi_{j,l,k}^{(d)}$  if  $\mu \in M$ , and  $s_\mu = \varphi_\mu = \varphi_{j_0,k}$  if  $\mu \in N$ .

**Lemma 2.2** *Let  $K$  satisfy  $(K^* K f)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$  and  $\{s_\mu, \mu \in \mathcal{M}\}$  be shearlets introduced in the first section. Define  $\hat{\sigma}_\mu(\xi) = (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)$  and  $U_\mu := 2^{-2\alpha j} K \sigma_\mu$ . Then  $\|U_\mu\| \leq C$  and for  $\mu \in \mathcal{M}$ ,*

$$\langle f, s_\mu \rangle = 2^{2\alpha j} \langle K f, U_\mu \rangle.$$

*Proof* By the Plancherel formula and the assumption  $\hat{\sigma}_\mu(\xi) = (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)$ , one knows that  $\langle f, s_\mu \rangle = \langle \hat{f}, \hat{s}_\mu \rangle = \langle \hat{f}(\xi), (b + |\xi|^2)^{-\alpha} \hat{\sigma}_\mu(\xi) \rangle$ . Moreover,

$$\langle f, s_\mu \rangle = \langle \hat{f}(\xi), (K^* K \sigma_\mu)^\wedge(\xi) \rangle = \langle f, K^* K \sigma_\mu \rangle = \langle K f, K \sigma_\mu \rangle = 2^{2\alpha j} \langle K f, U_\mu \rangle$$

due to  $(K^* K f)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$  and  $U_\mu := 2^{-2\alpha j} K \sigma_\mu$ .

Next, one shows  $\|U_\mu\| \leq C$ . Note that  $\|K \sigma_\mu\|^2 = \langle K \sigma_\mu, K \sigma_\mu \rangle = \langle K^* K \sigma_\mu, \sigma_\mu \rangle = \langle (K^* K \sigma_\mu)^\wedge, \hat{\sigma}_\mu \rangle$ ,  $(K^* K \sigma_\mu)^\wedge = (b + |\xi|^2)^{-\alpha} \hat{\sigma}_\mu(\xi)$  and  $\hat{\sigma}_\mu(\xi) = (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)$ . Then  $\|K \sigma_\mu\|^2 = \langle \hat{s}_\mu(\xi), (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi) \rangle$  and

$$\|U_\mu\|^2 = 2^{-4\alpha j} \|K \sigma_\mu\|^2 = 2^{-4\alpha j} \int_{\mathbb{R}^2} (b + |\xi|^2)^\alpha |\hat{s}_\mu(\xi)|^2 d\xi.$$

Because  $\text{supp } \hat{s}_\mu \subseteq C_j := [-2^{2j-1}, 2^{2j-1}]^2 \setminus [-2^{2j-4}, 2^{2j-4}]^2$ , one receives  $\|U_\mu\|^2 = 2^{-4\alpha j} \int_{C_j} (b + |\xi|^2)^\alpha |\hat{s}_\mu(\xi)|^2 d\xi \leq C$ . This completes the proof of Lemma 2.2.  $\square$

At the end of this section, we introduce two theorems which are important for our discussions. As in [8], we use  $STAR^2(A)$  to denote all sets  $B \subseteq [0, 1]^2$  with  $C^2$  boundary  $\partial B$  given by

$$\beta(\theta) = \begin{pmatrix} \rho(\theta) \cos \theta \\ \rho(\theta) \sin \theta \end{pmatrix}$$

in a polar coordinate system. Here,  $\rho(\theta) \leq \rho_0 < 1$  and  $|\rho''(\theta)| \leq A$ . Define  $\varepsilon^2(A) := \{f = f_0 + f_1 X_B, B \in STAR^2(A)\}$ , where  $f_0, f_1 \in C_0^\infty([0, 1]^2)$  are compactly supported on  $[0, 1]^2$ . Let

$c_\mu := \langle f, s_\mu \rangle$ ,  $M_j := \{(j, l, k, d), |k| \leq 2^{2j+1}, -2^j \leq l \leq 2^j - 1, d = 0, 1\}$  and

$$R(j, \varepsilon) = \{\mu \in M_j : |c_\mu| > \varepsilon\}.$$

Then with  $\sharp R(j, \varepsilon)$  standing for the cardinality of  $R(j, \varepsilon)$ , the following conclusion holds [8].

**Theorem 2.3** For  $f \in \varepsilon^2(A)$ ,  $\sharp R(j, \varepsilon) \leq C\varepsilon^{-\frac{2}{3}}$  and

$$\sum_{\mu \in M_j} |c_\mu|^2 \leq C2^{-2j}.$$

**Theorem 2.4** [14] Let  $X \sim N(u, 1)$  and  $t = \sqrt{2 \log(\eta^{-1})}$  with  $0 < \eta \leq \frac{1}{2}$ . Then

$$E|T_s(X, t) - u|^2 = [2 \log(\eta^{-1}) + 1](\eta + \min\{u^2, 1\}),$$

where  $N(u, 1)$  denotes the normal distribution with mean  $u$  and variance 1, while  $T_s(y, t) := \text{sgn}(y)(|y| - t)_+$  is the soft thresholding function.

### 3 Main theorem

In this section, we give an approximation result, which extends the result [8, Theorem 4.2] from the Radon transform to a family of linear operators. To do that, we introduce a set  $\mathcal{N}(\varepsilon)$  of significant shearlet coefficients as follows. Let

$$s_1 = \frac{1}{\frac{9}{2} + 6\alpha} \log_2(\varepsilon^{-1}), \quad s_2 = \frac{1}{\frac{3}{2} + 2\alpha} \log_2(\varepsilon^{-1}),$$

and  $j_0 = \lceil s_1 \rceil$ ,  $j_1 = \lceil s_2 \rceil$ . Define  $\mathcal{N}(\varepsilon) := M(\varepsilon) \cup N(\varepsilon) \subseteq \mathcal{M}$ , where

$$N(\varepsilon) = \{\mu = k \in \mathbb{Z}^2 : |k| \leq 2^{2j_0+1}\};$$

$$M(\varepsilon) = \{\mu = (j, l, k, d) : j_0 \leq j \leq j_1, -2^j \leq l \leq 2^j - 1, |k| \leq 2^{2j+1}, d = 0, 1\}.$$

Consider the model  $Y = Kf + \varepsilon W$  with  $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha} \hat{f}(\xi)$ . Lemma 2.2 tells that  $y_\mu := 2^{2\alpha j} \langle Y, U_\mu \rangle = \langle f, s_\mu \rangle + \varepsilon 2^{2\alpha j} n_\mu$ , where  $n_\mu$  is Gaussian noise with zero mean and bounded variance  $\sigma_\mu^2 = \|U_\mu\|^2 \leq C$  [15]. Let  $c_\mu = \langle f, s_\mu \rangle$  and  $\tilde{f} = \sum_{\mu \in \mathcal{N}(\varepsilon)} \tilde{c}_\mu s_\mu$  with

$$\tilde{c}_\mu = \begin{cases} T_s(y_\mu, \varepsilon \sqrt{2 \log(\sharp \mathcal{N}(\varepsilon))} 2^{2\alpha j} \sigma_\mu), & \mu \in \mathcal{N}(\varepsilon); \\ 0, & \text{otherwise,} \end{cases}$$

where  $T_s(y, t)$  is the soft thresholding function. Then the following result holds.

**Theorem 3.1** Let  $f \in \varepsilon^2(A)$  be the solution to  $Y = Kf + \varepsilon W$  with  $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha} \hat{f}(\xi)$  and  $\tilde{f}$  be defined as above. Then

$$\sup_{f \in \varepsilon^2(A)} E\|\tilde{f} - f\|^2 \leq C \log(\varepsilon^{-1}) \varepsilon^{\frac{2}{\frac{3}{2} + 2\alpha}} \quad (\varepsilon \rightarrow 0).$$

Here and in what follows,  $E$  stands for the expectation operator.

*Proof* Since  $\{s_\mu, \mu \in \mathcal{M}\}$  is a Parseval frame,  $f = \sum_{\mu \in \mathcal{M}} c_\mu s_\mu$  and  $\tilde{f} = \sum_{\mu \in \mathcal{N}(\varepsilon)} \tilde{c}_\mu s_\mu$ ,  $\tilde{f} - f = \sum_{\mu \in \mathcal{M}} (\tilde{c}_\mu - c_\mu) s_\mu$ . Moreover,  $\|\tilde{f} - f\|^2 = \sum_{\mu \in \mathcal{M}} |\tilde{c}_\mu - c_\mu|^2$  and

$$E\|\tilde{f} - f\|^2 = \sum_{\mu \in \mathcal{N}(\varepsilon)} E|\tilde{c}_\mu - c_\mu|^2 + \sum_{\mu \in \mathcal{N}(\varepsilon)^C} |c_\mu|^2. \quad (3.1)$$

In order to estimate  $\sum_{\mu \in \mathcal{N}(\varepsilon)^C} |c_\mu|^2$ , one observes  $\sum_{\mu \in M_j} |c_\mu|^2 \leq C2^{-2j}$  due to Theorem 2.3.

Then  $\sum_{j>j_1} \sum_{\mu \in M_j} |c_\mu|^2 \leq C \sum_{j>j_1} 2^{-2j} \leq C2^{-2j_1}$ . By  $2^{j_1} \lesssim \varepsilon^{-\frac{1}{\frac{3}{2}+2\alpha}}$ ,

$$\sum_{j>j_1} \sum_{\mu \in M_j} |c_\mu|^2 \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}. \quad (3.2)$$

(Here and in what follows,  $A \lesssim B$  denotes  $A \leq CB$  for some constant  $C > 0$ ).

Next, one considers  $c_\mu$  for  $j_0 \leq j \leq j_1$  and  $|k| \geq 2^{2j+1}$ . Note that  $|\psi^{(d)}(x)| \leq C_m(1 + |x|)^{-m}$  ( $d = 0, 1, m = 1, 2, \dots$ ). Then  $|\psi_{j,l,k}^{(d)}(x)| \leq C_m 2^{\frac{3}{2}j} (1 + |B_d^l A_d^j x - k|)^{-m}$ . Since  $f \in \varepsilon^2(A)$ ,  $\text{supp } f \subset Q_0 := [0, 1]^2$  and

$$|\langle f, \psi_{j,l,k}^{(d)} \rangle| \leq C_m 2^{\frac{3}{2}j} \|f\|_\infty \int_{Q_0} (1 + |B_d^l A_d^j x - k|)^{-m} dx.$$

On the other hand,  $|B_d^l A_d^j x| \leq \|B_d^l A_d^j\| |x| \leq 2^{2j} |x| \leq \sqrt{2} 2^{2j}$  for  $x \in Q_0$ . Hence,  $(1 + |B_d^l A_d^j x - k|)^{-m} \leq (1 + |k| - |B_d^l A_d^j x|)^{-m} \leq (|k| - \sqrt{2} 2^{2j})^{-m}$  for  $|k| \geq 2^{2j+1}$ . Moreover,  $\sum_{|k| \geq 2^{2j+1}} |c_\mu|^2 \leq 2^{3j} \sum_{|k| \geq 2^{2j+1}} (|k| - \sqrt{2} 2^{2j})^{-2m} = 2^{3j} \sum_{n=1}^{\infty} \sum_{2^{2j+n} \leq |k| \leq 2^{2j+n+1}} 2^{-4mj} (2^n - \sqrt{2})^{-2m} \lesssim 2^{3j} 2^{-4mj} \times \sum_{n=1}^{\infty} 2^{2(2j+n+1)} (2^n - \sqrt{2})^{-2m} \lesssim 2^{3j} 2^{-2j(2m-2)}$ , since  $m$  can be chosen big enough. Therefore,

$$\sum_{j=j_0}^{j_1} \sum_{l=-2^j}^{2^j-1} \sum_{|k| \geq 2^{2j+1}} |c_\mu|^2 \leq C_m \sum_{j=j_0}^{\infty} 2^{8j} 2^{-4mj} \lesssim 2^{-8j_0} \leq 2^{-6j_0} \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}} \quad (3.3)$$

due to the choice of  $j_0$ . The similar (even simpler) arguments show  $\sum_{|k| \geq 2^{2j_0+1}} |\langle f, \varphi_{j_0,k} \rangle|^2 \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}$  with  $\varphi_{j_0,k}(x) = 2^{j_0} \varphi(2^{j_0} x - k)$ . This with (3.2) and (3.3) leads to

$$\sum_{\mu \in \mathcal{N}(\varepsilon)^C} |c_\mu|^2 \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}. \quad (3.4)$$

Finally, one estimates  $\sum_{\mu \in \mathcal{N}(\varepsilon)} E|\tilde{c}_\mu - c_\mu|^2$ . By the definition of  $y_\mu$ ,  $\varepsilon^{-1} 2^{-2\alpha j} \sigma_\mu^{-1} y_\mu \sim N(\varepsilon^{-1} 2^{-2\alpha j} \sigma_\mu^{-1} c_\mu, 1)$ . Applying Theorem 2.4 with  $\eta^{-1} = \sharp \mathcal{N}(\varepsilon)$ , one obtains that

$$\begin{aligned} & E \left| T_s \left[ \varepsilon^{-1} 2^{-2\alpha j} \sigma_\mu^{-1} y_\mu, \sqrt{2 \log(\sharp \mathcal{N}(\varepsilon))} \right] - \varepsilon^{-1} 2^{-2\alpha j} \sigma_\mu^{-1} c_\mu \right|^2 \\ &= \left[ 2 \log(\sharp \mathcal{N}(\varepsilon)) + 1 \right] \left[ \frac{1}{\sharp \mathcal{N}(\varepsilon)} + \min \{ \varepsilon^{-2} 2^{-4\alpha j} \sigma_\mu^{-2} c_\mu^2, 1 \} \right]. \end{aligned}$$



Hence,  $E|T_s[y_\mu, \varepsilon 2^{2\alpha j} \sigma_\mu \sqrt{2 \log(\#\mathcal{N}(\varepsilon))}] - c_\mu|^2 \lesssim [2 \log(\#\mathcal{N}(\varepsilon)) + 1][\varepsilon^2 \frac{2^{4\alpha j} \sigma_\mu^2}{\#\mathcal{N}(\varepsilon)} + \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j} \sigma_\mu^2\}]$ . By  $\tilde{c}_\mu = T_s[y_\mu, \varepsilon 2^{2\alpha j} \sigma_\mu \sqrt{2 \log(\#\mathcal{N}(\varepsilon))}]$  for  $\mu \in \mathcal{N}(\varepsilon)$ , one knows that

$$E \sum_{\mu \in \mathcal{N}(\varepsilon)} |\tilde{c}_\mu - c_\mu|^2 \lesssim [2 \log(\#\mathcal{N}(\varepsilon)) + 1] \times \left[ \varepsilon^2 \sum_{\mu \in \mathcal{N}(\varepsilon)} \frac{2^{4\alpha j} \sigma_\mu^2}{\#\mathcal{N}(\varepsilon)} + \sum_{\mu \in \mathcal{N}(\varepsilon)} \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j} \sigma_\mu^2\} \right]. \quad (3.5)$$

Note that  $\mathcal{N}(\varepsilon) \cap M_j \subset \{(j, l, k, d) : |k| \leq 2^{2j+1}, |l| \leq 2^j\}$ . Then  $\#\mathcal{N}(\varepsilon) \leq C \sum_{j \leq j_1} 2^{5j} \lesssim 2^{5j_1} \lesssim \varepsilon^{-\frac{5}{\frac{3}{2}+2\alpha}}$ , and  $\log(\#\mathcal{N}(\varepsilon)) \lesssim \frac{10}{\frac{3}{2}+2\alpha} \log(\varepsilon^{-1}) \lesssim \log(\varepsilon^{-1})$ . Since  $\{\sigma_\mu : \mu \in \mathcal{M}\}$  is uniformly bounded,  $[2 \log(\#\mathcal{N}(\varepsilon)) + 1] \varepsilon^2 \sum_{\mu \in \mathcal{N}(\varepsilon)} \frac{2^{4\alpha j} \sigma_\mu^2}{\#\mathcal{N}(\varepsilon)} \lesssim \log(\varepsilon^{-1}) \varepsilon^2 2^{4\alpha j_1}$ . This with the choice of  $2^{j_1}$  shows that

$$[2 \log(\#\mathcal{N}(\varepsilon)) + 1] \varepsilon^2 \sum_{\mu \in \mathcal{N}(\varepsilon)} \frac{2^{4\alpha j} \sigma_\mu^2}{\#\mathcal{N}(\varepsilon)} \lesssim \log(\varepsilon^{-1}) \varepsilon^{\frac{3}{2}+2\alpha} \lesssim \log(\varepsilon^{-1}) \varepsilon^{\frac{2}{2}+2\alpha}. \quad (3.6)$$

It remains to estimate  $[2 \log(\#\mathcal{N}(\varepsilon)) + 1] \sum_{\mu \in \mathcal{N}(\varepsilon)} \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j} \sigma_\mu^2\}$ . Clearly,

$$\sum_{\mu \in \mathcal{N}(\varepsilon)} \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\} = \sum_{\{\mu \in \mathcal{N}(\varepsilon) : |c_\mu| \geq 2^{2\alpha j} \varepsilon\}} 2^{4\alpha j} \varepsilon^2 + \sum_{\{\mu \in \mathcal{N}(\varepsilon) : |c_\mu| < 2^{2\alpha j} \varepsilon\}} |c_\mu|^2. \quad (3.7)$$

By Theorem 2.3,  $\sum_{\{\mu \in M_j : |c_\mu| \geq 2^{2\alpha j} \varepsilon\}} 2^{4\alpha j} \varepsilon^2 \lesssim (2^{2\alpha j} \varepsilon)^{-\frac{2}{3}} 2^{4\alpha j} \varepsilon^2 \lesssim 2^{\frac{8}{3}\alpha j} \varepsilon^{\frac{4}{3}}$ . Hence,

$$\sum_{\{\mu \in \mathcal{N}(\varepsilon) : |c_\mu| \geq 2^{2\alpha j} \varepsilon\}} 2^{4\alpha j} \varepsilon^2 = \sum_{j=j_0}^{j_1} \sum_{\{\mu \in M_j : |c_\mu| \geq 2^{2\alpha j} \varepsilon\}} 2^{4\alpha j} \varepsilon^2 \lesssim 2^{\frac{8}{3}\alpha j_1} \varepsilon^{\frac{4}{3}}. \quad (3.8)$$

On the other hand,  $\sum_{\{\mu \in \mathcal{N}(\varepsilon) : |c_\mu| < 2^{2\alpha j} \varepsilon\}} |c_\mu|^2 = \sum_{j=j_0}^{j_1} \sum_{n=0}^{\infty} \sum_{\{2^{2\alpha j-n-1} \varepsilon < |c_\mu| \leq 2^{2\alpha j-n} \varepsilon\}} |c_\mu|^2$ . According to Theorem 2.3,  $\#\mathcal{R}(j, 2^{2\alpha j-n-1} \varepsilon) \lesssim 2^{-\frac{2}{3}(2\alpha j-n-1)} \varepsilon^{-\frac{2}{3}}$  and

$$\sum_{\{2^{2\alpha j-n-1} \varepsilon < |c_\mu| \leq 2^{2\alpha j-n} \varepsilon\}} |c_\mu|^2 \lesssim 2^{-\frac{2}{3}(2\alpha j-n-1)} \varepsilon^{-\frac{2}{3}} 2^{2(2\alpha j-n)} \varepsilon^2 \lesssim 2^{\frac{8}{3}\alpha j} 2^{-\frac{4}{3}n} 2^{\frac{2}{3}} \varepsilon^{\frac{4}{3}}.$$

Therefore,

$$\sum_{\{\mu \in \mathcal{N}(\varepsilon) : |c_\mu| < 2^{2\alpha j} \varepsilon\}} |c_\mu|^2 \leq \sum_{j=j_0}^{j_1} \sum_{n=0}^{\infty} 2^{\frac{8}{3}\alpha j} 2^{-\frac{4}{3}n} \varepsilon^{\frac{4}{3}} \leq 2^{\frac{8}{3}\alpha j_1} \varepsilon^{\frac{4}{3}}.$$

Combining this with (3.7) and (3.8), one knows that  $\sum_{\mu \in \mathcal{N}(\varepsilon)} \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\} \lesssim 2^{\frac{8}{3}\alpha j_1} \varepsilon^{\frac{4}{3}}$ . Furthermore,

$$\sum_{\mu \in \mathcal{N}(\varepsilon)} \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\} \lesssim \varepsilon^{\frac{2}{2\alpha+\frac{3}{2}}} \quad (3.9)$$

thanks to  $2^{\frac{1}{2}} \lesssim \varepsilon^{-\frac{1}{2\alpha+\frac{3}{2}}}$ . Now, it follows from (3.5), (3.6) and (3.9) that

$$\sum_{\mu \in \mathcal{N}(\varepsilon)} E|\tilde{c}_\mu - c_\mu|^2 \lesssim \log(\varepsilon^{-1}) \varepsilon^{\frac{2}{2\alpha+\frac{3}{2}}}.$$

This with (3.1) and (3.4) leads to the desired conclusion  $\sup_{f \in \varepsilon^2(A)} E\|\tilde{f} - f\|^2 \leq C \log(\varepsilon^{-1}) \times \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}$ . The proof is completed.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

LH and YL finished this work together. Two authors read and approved the final manuscript.

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